

LIMITS FOR CIRCULAR JACOBI BETA-ENSEMBLES

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ABSTRACT. Bourgade, Nikeghbali and Rouault recently proposed a matrix model for the circular Jacobi β -ensemble, which is a generalization of the Dyson circular β -ensemble but equipped with an additional parameter b , and further studied its limiting spectral measure. We calculate the scaling limits for expected products of characteristic polynomials of circular Jacobi β -ensembles. For the fixed constant b , the resulting limit near the spectrum singularity is proven to be a new multivariate function. When $b = \beta Nd/2$, the scaling limits in the bulk and at the soft edge agree with those of the Hermite (Gaussian), Laguerre (Chiral) and Jacobi β -ensembles proved in [17]. As corollaries, for even β the scaling limits of point correlation functions for the ensemble are given. Besides, a transition from the spectrum singularity to the soft edge limit is observed as b goes to infinity. The positivity of two special multivariate hypergeometric functions, which appear as one factor of the joint eigenvalue densities for spiked Jacobi/Wishart β -ensembles and Gaussian β -ensembles with source, will also be shown.

1. INTRODUCTION

1.1. Circular Jacobi ensembles. The circular unitary ensemble in random matrix theory usually refers to the unitary group $U(N)$ with its invariant Haar measure. The correlation functions and moments of characteristic polynomials for eigenvalues of these random unitary matrices are predicted to be somehow related with statistical properties of the Riemann zeta function in number theory [39]. Together with circular orthogonal and symplectic ensembles (see [29, Chap. 2] for definition and relationship to compact symmetric spaces) they consist Dyson three-fold way for circular ensembles. Dyson also observed that the induced eigenvalue densities correspond to the Gibbs measure of Coulomb log-gases on the circle at three different inverse temperatures $\beta = 1, 2, 4$. For general $\beta > 0$ these are called β -ensembles in the literature.

Inspired by Dumitriu-Edelman tridiagonal matrix models for Gaussian and Laguerre β -ensembles [22], Killip and Nenciu constructed analog matrix models for Dyson circular β -ensembles and also for Jacobi β -ensembles [41]. Combining circular and Jacobi ensembles, a more general ensemble was defined as the circular Jacobi ensemble in [29, 48]. Explicitly, the circular Jacobi ensemble refers to the probability density function on $[0, 2\pi)^N$

$$P_{b,N}(\theta_1, \dots, \theta_N) = \frac{1}{M_N(\bar{b}, b; \beta/2)} \prod_{j=1}^N \left(e^{i\frac{\bar{b}-b}{2}(\theta_j - \pi)} |1 - e^{i\theta_j}|^{\bar{b}+b} \right) |\Delta_N(e^{i\theta})|^\beta \quad (1.1)$$

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with $\text{Re}\{b\} > -1/2$, see [29, Chap. 3] for detailed discussion and [12] for the matrix model realization. Here the constant $M_N(a', b'; \alpha)$ equals to (see, e.g., [29, Chap. 4])

$$M_N(a', b'; \alpha) = (2\pi)^N \prod_{j=0}^{N-1} \frac{\Gamma(1 + \alpha + j\alpha)\Gamma(1 + a' + b' + j\alpha)}{\Gamma(1 + \alpha)\Gamma(1 + a' + j\alpha)\Gamma(1 + b' + j\alpha)}. \quad (1.2)$$

and the Vandermonde determinant

$$\Delta_N(e^{i\theta}) = \prod_{1 \leq j < k \leq N} (e^{i\theta_k} - e^{i\theta_j}).$$

For $\beta = 2$, such measures were first studied by Hua [33] and later by Nagao [48] motivated by random matrix theory. For $\beta = 1, 2, 4$ the point correlation functions and their scaling limits were investigated by Forrester and Nagao [31]. Note that for $b = 0$ (1.1) reduces to the density for the circular ensemble and for $b \in \frac{\beta}{2}\mathbb{N}$ it coincides with the circular ensemble given that there is an eigenvalue at 1. For general b it is said to describe a spectrum singularity.

Applying an appropriate change of variables we can derive some classical random matrix ensembles from (1.1). For instance, with

$$e^{i\theta_j} = \frac{1 + i\lambda_j}{1 - i\lambda_j}, \quad j = 1, 2, \dots, N,$$

(1.1) reduces to the Cauchy ensemble, which shows that both ensembles are in fact equivalent, see Chapter 3.9, [29]. Another example is that the Laguerre ensemble can also be treated as a limit transition from (1.1). The latter fact is a remark of E.M. Rains (AIM workshop 2009) communicated to the author by P.J. Forrester, which can be formally inferred as follows. Let $c = \bar{b} + b$ be a fixed constant and let $\text{Im}\{b\} = i\frac{\bar{b}-b}{2}$ go to $-\infty$, with change of variables

$$\theta_j = -\frac{x_j}{\text{Im}\{b\}}, \quad j = 1, 2, \dots, N,$$

we see that

$$P_{b,N}(\theta) d^N \theta \sim \frac{1}{Z_N} \prod_{j=1}^N x_j^c e^{-x_j} |\Delta_N(x)|^\beta d^N x. \quad (1.3)$$

Here all $x_j \in (0, \infty)$ and

$$Z_N = M_N(\bar{b}, b; \beta/2) e^{\pi N \text{Im}\{b\}} (-\text{Im}\{b\})^{(c+1)N + \beta N(N-1)/2} \quad (1.4)$$

goes to the normalization constant for the Laguerre ensemble (see (3.129) in [29] for the explicit constant) as $\text{Im}\{b\} \rightarrow -\infty$ with the help of Stirling formulas.

1.2. Limiting spectral measure. The empirical spectral distribution for the circular Jacobi ensemble refers to

$$\mu_N = \frac{1}{N} \sum_{k=1}^N \delta_{e^{i\theta_k}}. \quad (1.5)$$

With $b = b_N = \beta N d/2$ where $\text{Re}\{d\} \geq 0$, as $N \rightarrow \infty$ Bourgade etc. [12] have proved that μ_N converges weakly in probability to some limit $\mu_d(\zeta)$ ($\zeta = e^{i\theta}$). Jokela etc. [37] gave another alternative derivation from the viewpoint of the equilibrium measure. To give the explicit form of $\mu_d(\zeta)$, we need some more notations.

Let $\theta_d \in [0, \pi)$ and $\xi_d \in [-\theta_d, \theta_d]$ be such that

$$\sin \frac{\theta_d}{2} = \left| \frac{d}{1+d} \right|, \quad e^{i\xi_d} = \frac{1+d}{1+\bar{d}}. \quad (1.6)$$

If $\operatorname{Re}\{d\} \geq 0, d \neq 0$, let ω_d be defined by

$$\omega_d(\theta) = \begin{cases} \frac{\sqrt{\sin^2((\theta-\xi_d)/2) - \sin^2(\theta_d/2)}}{|1/(1+d)| \sin(\theta/2)}, & \theta \in (\theta_d + \xi_d, 2\pi - \theta_d + \xi_d) \\ 0, & \text{otherwise.} \end{cases} \quad (1.7)$$

In this case, let

$$d\mu_d(\zeta) := \omega_d(\theta) \frac{d\theta}{2\pi} \quad (\zeta = e^{i\theta}), \quad (1.8)$$

and for $d = 0$, let

$$d\mu_0(\zeta) := \frac{d\theta}{2\pi} \quad (\zeta = e^{i\theta}), \quad (1.9)$$

the Haar measure on the unit circle.

In particular, if $d \in \mathbb{R}$ and $d \geq 0$, then $\xi_d = 0$. For $d > 0$, $\theta = \theta_d$ (or $2\pi - \theta_d$) is the so-called soft edge of the spectrum while the interval $(\theta_d, 2\pi - \theta_d)$ is the bulk of the spectrum. In this case, we will investigate the local limits for correlations of characteristic polynomials in the circular Jacobi ensembles at the edge and in the bulk as in the Hermite (Gaussian), Laguerre (Chiral) and Jacobi β -ensembles of [17].

1.3. Correlations. For the characteristic polynomial of the circular Jacobi ensemble denoted by

$$\phi(z) = \prod_{j=1}^N (1 - ze^{i\theta_j}) \quad (1.10)$$

and its conjugate by

$$\overline{\phi(z)} = \prod_{j=1}^N (1 - \bar{z}e^{-i\theta_j}), \quad (1.11)$$

our aim is to prove local limits of the expectation of products of characteristic polynomials

$$K_{b,N}(s; t) = \mathbb{E} \left[\prod_{j=1}^m \phi(s_j) \prod_{k=1}^n \overline{\phi(t_k)} \right], \quad (1.12)$$

as $N \rightarrow \infty$. Here $\mathbb{E}[f]$ means the expectation of f under the circular Jacobi ensemble.

At this point, it is worth stressing that the above average $K_{b,N}(s; t)$ allows the evaluation of k -point correlation functions for even β . Suppose for convenience that there are $k + N$ eigenvalues denoted by $e^{ir_1}, \dots, e^{ir_k}, e^{i\theta_1}, \dots, e^{i\theta_N}$ in the circular Jacobi ensemble, or equivalently the density function of (1.1) is replaced by $P_{b,k+N}(r_1, \dots, r_k, \theta_1, \dots, \theta_N)$. Accordingly, the k -point correlation function [29, 47]

is specified as a canonical average by

$$R_{b,N}^{(k)}(r_1, \dots, r_k) = \frac{(k+N)!}{N!} \frac{1}{M_{k+N}(\bar{b}, b; \beta/2)} |\Delta_k(e^{ir})|^\beta \prod_{j=1}^k e^{i\frac{\bar{b}-b}{2}(r_j-\pi)} |1-e^{ir_j}|^{\bar{b}+b} \\ \times \int \prod_{l=1}^k \prod_{j=1}^N |e^{ir_l} - e^{i\theta_j}|^\beta \prod_{j=1}^N e^{i\frac{\bar{b}-b}{2}(\theta_j-\pi)} |1-e^{i\theta_j}|^{\bar{b}+b} |\Delta_N(e^{i\theta})|^\beta d^N \theta. \quad (1.13)$$

Therefore for even β , if we let $m = \beta k/2$ and introduce

$$s_{l+\beta(j-1)/2} = e^{-ir_j}, \quad j = 1, \dots, k, l = 1, \dots, \beta/2, \quad (1.14)$$

then it is easy to verify that

$$R_{b,N}^{(k)}(r_1, \dots, r_k) = \frac{(k+N)!}{N!} \frac{M_N(\bar{b}, b; \beta/2)}{M_{k+N}(\bar{b}, b; \beta/2)} |\Delta_k(e^{ir})|^\beta \\ \times \prod_{j=1}^k e^{i\frac{\bar{b}-b}{2}(r_j-\pi)} |1-e^{ir_j}|^{\bar{b}+b} K_{b,N}(s; s). \quad (1.15)$$

1.4. Goal and organization of the paper. We consider the expectation of products of characteristic polynomials (1.12) for the circular Jacobi β -ensemble defined by (1.1). Our aim is to prove the local limit for the expectation as $N \rightarrow \infty$ and to give explicit expression of the limit function in terms of Jack polynomials for any $\beta > 0$. As a direct application, for even β we prove the local limit of correlation functions. The similar results have been shown for other three classical β -ensembles in random matrix theory, namely the Hermite (Gaussian), Laguerre (Chiral), and Jacobi β -ensembles [17].

For special values of the Dyson index $\beta = 1, 2, 4$, we recover more conventional random matrix ensembles, see [29, 47]. Indeed, for $\beta = 1, 2, 4$, the scaling limits for ratios and products of characteristic polynomials are already well established, see [3, 5, 7, 11, 34] and references therein.

In the present article we deal with the general β case. For the Gaussian β -ensembles, Aomoto [4] and more recently Su [56] proved the scaling limit for the product of $n = 2$ characteristic polynomials. In the previous paper [17], we (with P. Desrosiers) completely proved the scaling limit for products of arbitrary n characteristic polynomials in the Hermite (Gaussian), Laguerre (Chiral), and Jacobi β -ensembles. Other properties concerning β -ensembles, for instance, duality and Jack polynomials [6, 15, 16, 18, 25, 28, 46], stochastic differential equations [26, 51, 52], the global fluctuations [8, 23, 24, 40], the gap probabilities and the largest eigenvalues [21, 32, 36, 52, 53, 58, 59], were also studied in the limit $N \rightarrow \infty$. More recently, some universality results concerning the general $\beta > 0$ and general potential case have been obtained [9, 10]. Much progress has been made for the circular Jacobi ensembles, see [13, 14, 31, 45, 50, 55].

The rest of this paper is organized as follows. In Section 2 we briefly review some aspects of Jack polynomials and associated hypergeometric functions. The positivity of two special functions ${}_1\mathcal{F}_0^{(\alpha)}$ and ${}_0\mathcal{F}_0^{(\alpha)}$ will be shown. In Sections 3 and 4 we turn to calculations of the local limits of the bulk and edge respectively for the fixed constant b and for $b = \beta Nd/2$. In particular for the former a new

multivariate function is found to appear as a scaling limit near the spectrum singularity. In Section 5 we derive a limit transition from the spectrum singularity to the soft edge.

2. JACK POLYNOMIALS AND HYPERGEOMETRIC FUNCTIONS

This section provides a brief review of Jack polynomials and associated hypergeometric functions [38, 57, 61], see [29, Chap. 12] for more recent textbook treatments.

2.1. Partitions. A partition $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_i, \dots)$ is a sequence of non-negative integers κ_i such that

$$\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_i \geq \dots$$

and only a finite number of the terms κ_i are non-zero. The number of non-zero terms is referred to as the length of κ , and is denoted $\ell(\kappa)$. We shall not distinguish between two partitions that differ only by a string of zeros. The weight of a partition κ is the sum

$$|\kappa| := \kappa_1 + \kappa_2 + \dots$$

of its parts, and its diagram is the set of points $(i, j) \in \mathbb{N}^2$ such that $1 \leq j \leq \kappa_i$. Reflection in the diagonal produces the conjugate partition $\kappa' = (\kappa'_1, \kappa'_2, \dots)$.

The set of all partitions of a given weight is partially ordered by the dominance order: $\kappa \leq \sigma$ if and only if $\sum_{i=1}^k \kappa_i \leq \sum_{i=1}^k \sigma_i$ for all k .

2.2. Jack polynomials. Let $\Lambda_n(x)$ be the algebra of symmetric polynomials in n variables x_1, \dots, x_n with coefficients in the field $\mathbb{F} = \mathbb{Q}(\alpha)$, which is the field of rational functions in the parameter α . As a ring, $\Lambda_n(x)$ is generated by the power-sums:

$$p_k(x) := x_1^k + \dots + x_n^k. \quad (2.1)$$

The ring of symmetric polynomials is naturally graded: $\Lambda_n(x) = \bigoplus_{k \geq 0} \Lambda_n^k(x)$, where $\Lambda_n^k(x)$ denotes the set of homogeneous polynomials of degree k . As a vector space, $\Lambda_n^k(x)$ equals to the span over \mathbb{F} of all symmetric monomials $m_\kappa(x)$, where κ is a partition of weight k and

$$m_\kappa(x) := x_1^{\kappa_1} \dots x_n^{\kappa_n} + \text{distinct permutations.}$$

Note that if the length of the partition κ is larger than n , we set $m_\kappa(x) = 0$.

The whole ring $\Lambda_n(x)$ is invariant under the action of homogeneous differential operators related to the Calogero-Sutherland models [6]:

$$E_k = \sum_{i=1}^n x_i^k \frac{\partial}{\partial x_i}, \quad D_k = \sum_{i=1}^n x_i^k \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{1 \leq i \neq j \leq n} \frac{x_i^k}{x_i - x_j} \frac{\partial}{\partial x_i}, \quad k = 0, 1, 2, \dots \quad (2.2)$$

The operators E_1 and D_2 can be used to define the Jack polynomials. Indeed, for each partition κ , there exists a unique symmetric polynomial $P_\kappa^{(\alpha)}(x)$ that satisfies the following two conditions [57]:

$$(1) \quad P_\kappa^{(\alpha)}(x) = m_\kappa(x) + \sum_{\mu < \kappa} c_{\kappa\mu} m_\mu(x) \quad (\text{triangularity}) \quad (2.3)$$

$$(2) \quad \left(D_2 - \frac{2}{\alpha} (n-1) E_1 \right) P_\kappa^{(\alpha)}(x) = \epsilon_\kappa P_\kappa^{(\alpha)}(x) \quad (\text{eigenfunction}) \quad (2.4)$$

where $\epsilon_\kappa, c_{\kappa\mu} \in \mathbb{F}$. Because of the triangularity condition, $\Lambda_n(x)$ is also equal to the span over \mathbb{F} of all Jack polynomials $P_\kappa^{(\alpha)}(x)$, with $|\kappa| \leq n$.

2.3. Hypergeometric series. For $(i, j) \in \kappa$, let

$$a_\kappa(i, j) = \kappa_i - j \quad \text{and} \quad l_\kappa(i, j) = \kappa'_j - i. \quad (2.5)$$

Introduce the hook-length of κ defined by

$$h_\kappa^{(\alpha)} = \prod_{(i,j) \in \kappa} \left(1 + a_\kappa(i, j) + \frac{1}{\alpha} l_\kappa(i, j) \right), \quad (2.6)$$

and the α -deformation of the Pochhammer symbol by

$$[x]_\kappa^{(\alpha)} = \prod_{1 \leq i \leq \ell(\kappa)} \left(x - \frac{i-1}{\alpha} \right)_{\kappa_i}. \quad (2.7)$$

Here $(x)_j \equiv x(x+1) \cdots (x+j-1)$.

With the previous notations we now turn to the precise definition of the hypergeometric series associated with Jack polynomials, see e.g. [38, 61]. Fix $p, q \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and let $a_1, \dots, a_p, b_1, \dots, b_q$ be complex numbers such that $(i-1)/\alpha - b_j \notin \mathbb{N}_0$ for all $i \in \mathbb{N}_0$. The (p, q) -type hypergeometric series in two sets of n variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ reads off

$$\begin{aligned} {}_p\mathcal{F}_q^{(\alpha)}(a_1, \dots, a_p; b_1, \dots, b_q; x; y) = \\ \sum_{k=0}^{\infty} \sum_{\ell(\kappa) \leq n, |\kappa|=k} \frac{1}{h_\kappa^{(\alpha)}} \frac{[a_1]_\kappa^{(\alpha)} \cdots [a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)} \cdots [b_q]_\kappa^{(\alpha)}} \frac{P_\kappa^{(\alpha)}(x) P_\kappa^{(\alpha)}(y)}{P_\kappa^{(\alpha)}(1^{(n)})}, \end{aligned} \quad (2.8)$$

where the shorthand notation $1^{(n)}$ stands for $1, \dots, 1$ with n times and

$$P_\kappa^{(\alpha)}(1^{(n)}) = \prod_{(i,j) \in \kappa} \frac{n + \alpha(j-1) - (i-1)}{1 + \alpha a_\kappa(i, j) + l_\kappa(i, j)}, \quad (2.9)$$

see (10.20), [44]. In particular, when $y_1 = \dots = y_n = 1$, it reduces to the hypergeometric series

$${}_pF_q^{(\alpha)}(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{k=0}^{\infty} \sum_{|\kappa|=k} \frac{1}{h_\kappa^{(\alpha)}} \frac{[a_1]_\kappa^{(\alpha)} \cdots [a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)} \cdots [b_q]_\kappa^{(\alpha)}} P_\kappa^{(\alpha)}(x). \quad (2.10)$$

From now on we always assume $\alpha > 0$. Note that when $p \leq q$, the series (2.10) converges absolutely for all $x \in \mathbb{C}^n$, while for $p = q + 1$, (2.10) converges absolutely for all $|x_i| < r$ with some positive constant r , see [38, Proposition 1] for more details about the convergence issue. It immediately follows from the definition and the convergence of (2.10) that for $p \leq q$ (2.8) converges absolutely for all $x, y \in \mathbb{C}^n$, while for $p = q + 1$ (2.8) converges absolutely for all $|x_i y_j| < r$.

In the special case of $p = 1, q = 0$ we can say more about the convergence since

$${}_1F_0^{(\alpha)}(a; x) = \prod_{j=1}^n (1 - x_j)^{-a}, \quad (2.11)$$

see [61, Proposition 3.1]. Therefore, the series ${}_1F_0^{(\alpha)}(a; x)$ converges for all $|x_i| < 1$ and can be extended to all $\operatorname{Re}\{x_i\} < 1$ by analytic continuation. Furthermore, it is easy to show that ${}_1\mathcal{F}_0^{(\alpha)}(a; x; y)$ converges for all $|x_i y_j| < 1$. We will prove that the series ${}_1\mathcal{F}_0^{(\alpha)}(a; x; y)$ can be extended to all $\operatorname{Re}\{x_i y_j\} < 1$ by analytic continuation in the subsequent subsection.

When $\alpha = 1$ there exist closed-form expressions for the hypergeometric series (2.8) and (2.10).

Proposition 2.1 (Closed form for $\alpha = 1$).

$$\begin{aligned} {}_p\mathcal{F}_q^{(1)}(a_1 + n - 1, \dots, a_p + n - 1; b_1 + n - 1, \dots, b_q + n - 1; x; y) \\ = \prod_{i=0}^{n-1} \frac{i! (b_1)_i \cdots (b_q)_i}{(a_1)_i \cdots (a_p)_i} \frac{\det[{}_pf_q(x_j y_k)]}{\det[x_j^{k-1}] \det[y_j^{k-1}]}, \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} {}_pF_q^{(1)}(a_1 + n - 1, \dots, a_p + n - 1; b_1 + n - 1, \dots, b_q + n - 1; x) \\ = \prod_{i=0}^{n-1} \frac{i! (b_1)_i \cdots (b_q)_i}{(a_1)_i \cdots (a_p)_i} \frac{\det[x_j^{k-1} {}_pf_q^{(k-1)}(x_j)]}{\det[x_j^{k-1}]}. \end{aligned} \quad (2.13)$$

Here

$${}_pf_q(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}.$$

Proof. First note that for $\alpha = 1$ the Jack polynomial $P_{\kappa}^{(1)}(x)$ is equal to the Schur polynomial $s_{\kappa}^{(1)}(x)$ which is defined by

$$s_{\kappa}^{(1)}(x) = \frac{\det[x_j^{\kappa_{n+1-k} + k - 1}]}{\det[x_j^{k-1}]}.$$

Let

$$c_k = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{1}{k!},$$

by Theorem 1.2.2 [33], we have the expansion

$$\frac{\det[{}_pf_q(x_j y_k)]}{\det[x_j^{k-1}] \det[y_j^{k-1}]} = \sum_{\kappa} \prod_{i=1}^n c_{\kappa_i + n - i} P_{\kappa}^{(1)}(x) P_{\kappa}^{(1)}(y). \quad (2.14)$$

Since $\prod_{i=1}^n c_{\kappa_i + n - i}$ equals to

$$\prod_{i=1}^n \frac{(a_1)_{n-i} \cdots (a_p)_{n-i}}{(b_1)_{n-i} \cdots (b_q)_{n-i}} \frac{1}{(1)_{n-i}} \frac{[a_1 + n - 1]_{\kappa}^{(1)} \cdots [a_p + n - 1]_{\kappa}^{(1)}}{[b_1 + n - 1]_{\kappa}^{(1)} \cdots [b_q + n - 1]_{\kappa}^{(1)}} \frac{1}{[n]_{\kappa}^{(1)}}$$

and

$$h_{\kappa}^{(1)} P_{\kappa}^{(1)}(1^{(n)}) = [n]_{\kappa}^{(1)},$$

the equation (2.12) follows.

Let all y_j go to 1 and take the derivative we then obtain (2.13) from (2.12), or by Theorem 1.2.4 [33]. \square

2.4. Invariance property with application. To prove scaling limits for classical β -ensembles the translation invariance property of two special hypergeometric functions has been proven to be of vital importance (see [17, Proposition 2.2]) and will also play a key role in the present article. Explicitly, for complex numbers a, b and the variables $x = (x_1, \dots, x_n)$, let

$$(a^{(n)}) = (\overbrace{a, \dots, a}^n), \quad \frac{b + ax}{1 + cx} = \left(\frac{b + ax_1}{1 + cx_1}, \dots, \frac{b + ax_n}{1 + cx_n} \right),$$

then the following translation invariance holds

$${}_1\mathcal{F}_0^{(\alpha)}(a; b+x; y) = \prod_{j=1}^n (1-by_j)^{-a} {}_1\mathcal{F}_0^{(\alpha)}(a; x; \frac{y}{1-by}), \quad (2.15)$$

$${}_0\mathcal{F}_0^{(\alpha)}(a+x; b+y) = \exp\{nab + ap_1(y) + bp_1(x)\} {}_0\mathcal{F}_0^{(\alpha)}(x; y) \quad (2.16)$$

and further

$$\begin{aligned} & {}_0\mathcal{F}_0^{(\alpha)}(x_1, \dots, x_n; a^{(k)}, b^{(n-k)}) \\ &= e^{bp_1(x)} {}_1F_1^{(\alpha)}(k/\alpha; n/\alpha; (a-b)x_1, \dots, (a-b)x_n) \end{aligned} \quad (2.17)$$

$$= e^{(a-b)kx_1+bp_1(x)} {}_1F_1^{(\alpha)}(k/\alpha; n/\alpha; (a-b)(x_2-x_1), \dots, (a-b)(x_n-x_1)). \quad (2.18)$$

See Proposition 2.2 and Corollary 2.3 of [17] for the proofs of (2.15)–(2.18).

We now give three application examples of the translation formulas (2.15) and (2.16). The first one is concerned with analytic continuation of the series ${}_1\mathcal{F}_0^{(\alpha)}$.

Proposition 2.2. *The domain of the series ${}_1\mathcal{F}_0^{(\alpha)}(a; x; y)$ can be extended by analytic continuation to some region at least containing $x, y \in \mathbb{C}^n$ such that*

$$\operatorname{Re}\{x_i y_j\} < 1, \quad y_j \geq 0, \quad \forall i, j = 1, \dots, n. \quad (2.19)$$

Proof. We just deal with the points $x, y \in \mathbb{C}^n$ satisfying (2.19).

First, for $|x_i y_j| < 1$ ($i, j = 1, \dots, n$), the series converges absolutely, as discussed in Subsection 2.3. Otherwise, there exist x_i, y_j such that $\operatorname{Re}\{x_i y_j\} < 1$, $y_j \geq 0$ but $|x_i y_j| \geq 1$. Write

$$b_0 = \max \left\{ \frac{|x_i y_j|^2 - 1}{2y_j(1 - \operatorname{Re}\{x_i y_j\})} : 1 \leq i, j \leq n \right\},$$

then $b_0 \geq 0$.

Take $b > b_0$, it's easy to verify that

$$\left| (b+x_i) \frac{y_j}{1+by_j} \right| < 1, \quad \forall i, j = 1, \dots, n. \quad (2.20)$$

On the other hand, by (2.15)

$${}_1\mathcal{F}_0^{(\alpha)}(a; x; y) = \prod_{j=1}^n (1+by_j)^{-a} {}_1\mathcal{F}_0^{(\alpha)}(a; b+x; \frac{y}{1+by}), \quad (2.21)$$

therefore by (2.20) the RHS of (2.21) gives an analytic continuation. \square

The second one concerns the positivity of ${}_1\mathcal{F}_0^{(\alpha)}$ and ${}_0\mathcal{F}_0^{(\alpha)}$, which appear as one factor of the joint eigenvalue densities for spiked Jacobi/Wishart β -ensembles and Gaussian β -ensembles with source [15, 19, 20, 30, 60]. We remark that the positivity of ${}_0\mathcal{F}_0^{(\alpha)}$ has been proved by Rösler (see Corollary 4.9 [54]), but the proof here is quite different.

Proposition 2.3. *For all $x_i, y_j \in \mathbb{R}$ ($i, j = 1, \dots, n$),*

$${}_0\mathcal{F}_0^{(\alpha)}(x; y) > 0, \quad (2.22)$$

and for all $x_i y_j < 1, y_j \geq 0$ ($i, j = 1, \dots, n$) and $a \geq \frac{n-1}{\alpha}$,

$${}_1\mathcal{F}_0^{(\alpha)}(a; x; y) > 0. \quad (2.23)$$

Note that by Proposition 2.2 the function in (2.23) makes sense in the assumed region of the argument.

Proof. For (2.22), it holds when all $x_i, y_j \geq 0$ ($i, j = 1, \dots, n$) by definition, since the coefficients of Jack polynomials are nonnegative (see Theorem 1.1 [42]). Otherwise, if some $x_i < 0$ or $y_j < 0$, set

$$a = \max\{-x_1, \dots, -x_n, 0\}, \quad b = \max\{-y_1, \dots, -y_n, 0\},$$

then $a + x_k \geq 0$ and $b + y_l \geq 0$ for all k, l . By (2.16) we have

$${}_0\mathcal{F}_0^{(\alpha)}(x; y) = \exp\{-nab - ap_1(y) - bp_1(x)\} {}_0\mathcal{F}_0^{(\alpha)}(a + x; b + y) > 0.$$

For (2.23), it holds when all $x_i y_j < 1, x_i \geq 0, y_j \geq 0$ ($i, j = 1, \dots, n$), since all the coefficients are non-negative by definition. Otherwise, there exists some $x_i < 0$, set $b = \max\{-x_1, \dots, -x_n\}$, then $b > 0$ and $b + x_j \geq 0$ for all j . Since $(b + x_i) \frac{y_j}{1 + by_j} < 1$ by the assumption, application of (2.15) gives

$${}_1\mathcal{F}_0^{(\alpha)}(a; x; y) = \prod_{j=1}^n (1 + by_j)^{-a} {}_1\mathcal{F}_0^{(\alpha)}(a; b + x; \frac{y}{1 + by}) > 0.$$

□

The last one is concerned with Dubbs-Edelman β -MANOVA models with general covariance [19], which may also be called spiked β -Jacobi ensembles. The β -MANOVA model builds on the β -Wishart ensemble which is introduced in [20, 30, 60], see [19] for the detailed definition. However, for $\beta = 1, 2, 4$, it can be defined as follows, see [19] for the three cases and [35] for $\beta = 1$ case. Let X be an $m \times n$ matrix and Y be a $p \times n$ matrix with independent real, complex, quaternion normal variables, and let $\Sigma = \text{diag}\{\sigma_1, \dots, \sigma_n\}$ with $\sigma_j > 0$, the random matrix $\Lambda = Y^* Y (Y^* Y + \sqrt{\Sigma} X^* X \sqrt{\Sigma})^{-1}$ refers to the β -MANOVA model.

Let $\lambda_1, \dots, \lambda_n$ be generalized eigenvalues of β -MANOVA models (that is, the generalized eigenvalues of Λ when $\beta = 1, 2, 4$), Dubbs and Edelman have derived the joint density function for generalized eigenvalues [19, Theorem 1.1]. We here just give another expression of the density function after application of the equation (2.15) and Proposition 2.2. Our expression is more similar in form to that of the Jacobi β -ensemble.

Proposition 2.4. *Let $\lambda_1, \dots, \lambda_n$ be generalized eigenvalues of β -MANOVA models with covariance $\Sigma = \text{diag}\{\sigma_1, \dots, \sigma_n\}$ defined in [19], and let $p, m > n - 1$. Then the joint density function for eigenvalues is equal to*

$$\begin{aligned} P_{n,\beta}(\lambda) &= \frac{\mathcal{K}_{p+m,n}^{(\beta)}}{\mathcal{K}_{p,n}^{(\beta)} \mathcal{K}_{m,n}^{(\beta)}} \prod_{i=1}^n \lambda_i^{\frac{p-n+1}{2}\beta-1} (1 - \lambda_i)^{\frac{m-n+1}{2}\beta-1} |\Delta_n(\lambda)|^\beta \\ &\times {}_1\mathcal{F}_0^{(2/\beta)}\left(\frac{p+m}{2}\beta; \lambda; 1 - \sigma\right), \quad 0 < \lambda_1, \dots, \lambda_n < 1, \end{aligned} \quad (2.24)$$

where

$$\mathcal{K}_{m,n}^{(\beta)} = 2^{\beta mn/2} (\Gamma(\beta/2))^{-n} \prod_{i=1}^n \Gamma(\beta i/2) \Gamma(\beta(m - n + i)/2).$$

Note that the factor ${}_1\mathcal{F}_0^{(2/\beta)}$ of the joint density in (2.24) makes sense by Proposition 2.2 and is positive by Proposition 2.3. Also note that the notation ${}_1F_0^{(\beta)}$ is used in [19] rather than ${}_1\mathcal{F}_0^{(2/\beta)}$ here, but they refer to the same function.

Proof. Theorem 1.1 [19] shows the joint density function equals

$$P_{n\beta}(\lambda) = \frac{\mathcal{K}_{p+m,n}^{(\beta)}}{\mathcal{K}_{p,n}^{(\beta)} \mathcal{K}_{m,n}^{(\beta)}} \prod_{i=1}^n \lambda_i^{\frac{p-n+1}{2}\beta-1} (1-\lambda_i)^{-\frac{p+n-1}{2}\beta-1} |\Delta_n(\lambda)|^\beta \times {}_1\mathcal{F}_0^{(2/\beta)}\left(\frac{p+m}{2}\beta; -\frac{\lambda}{1-\lambda}; \sigma\right). \quad (2.25)$$

By (2.15) we have

$${}_1\mathcal{F}_0^{(2/\beta)}\left(\frac{p+m}{2}\beta; -\frac{\lambda}{1-\lambda}; \sigma\right) = {}_1\mathcal{F}_0^{(2/\beta)}\left(\frac{p+m}{2}\beta; \frac{\lambda}{1-\lambda}; -\sigma\right) \quad (2.26)$$

$$= \prod_{i=1}^n (1-\lambda_i)^{\frac{p+m}{2}\beta} {}_1\mathcal{F}_0^{(2/\beta)}\left(\frac{p+m}{2}\beta; \lambda; 1-\sigma\right), \quad (2.27)$$

with (2.25) which completes the proof. \square

3. SPECTRUM SINGULARITY

In this section we consider the case of the fixed constant b . The starting point is the following Selberg correlation integrals (see, e.g., [29, Proposition 13.1.2])

$$\int_{[0,2\pi)^N} |\Delta_N(e^{i\theta})|^{2/\alpha} \prod_{j=1}^N e^{i\frac{a'-b'}{2}(\theta_j-\pi)} |1 - e^{i\theta_j}|^{a'+b'} \prod_{j=1}^m \phi(s_j) d^N\theta \\ = M_N(a', b'; 1/\alpha) {}_2F_1^{(1/\alpha)}(-N, \alpha b'; -N+1-\alpha(1+a'); s_1, \dots, s_m). \quad (3.1)$$

We first give an expression of the expected products of characteristic polynomials in terms of the generalized hypergeometric functions, which will also be used later in the subsequent sections.

Proposition 3.1.

$$K_{b,N}(s; t) = C_N \prod_{k=1}^n (\overline{t_k})^N \times {}_2F_1^{(\beta/2)}(-N, (2/\beta)(b+n); (2/\beta)(\bar{b}+b+m+n); 1-s_1, \dots, 1-s_{m+n}), \quad (3.2)$$

where $K_{b,N}(s; t)$ is given in (1.12), $s_{m+j} = (\overline{t_j})^{-1}$ for $j = 1, \dots, n$ and

$$C_N = \prod_{j=0}^{m-1} \frac{\Gamma((2/\beta)(1+\bar{b}+j)) \Gamma(N+(2/\beta)(1+\bar{b}+b+n+j))}{\Gamma((2/\beta)(1+\bar{b}+b+n+j)) \Gamma(N+(2/\beta)(1+b+j))} \\ \times \prod_{j=0}^{n-1} \frac{\Gamma((2/\beta)(1+b+j)) \Gamma(N+(2/\beta)(1+\bar{b}+b+j))}{\Gamma((2/\beta)(1+\bar{b}+b+j)) \Gamma(N+(2/\beta)(1+b+j))}. \quad (3.3)$$

Proof. Since

$$\overline{\phi(z)} = (\bar{z})^N \phi(1/\bar{z}) \prod_{j=1}^N e^{-i(\theta_j-\pi)},$$

set $s_{m+j} = (\bar{t}_j)^{-1}$, by (3.1) with $a' = \bar{b} - n$, $b' = b + n$ and $2/\alpha = \beta$, we get

$$K_{b,N}(s; t) = \prod_{k=1}^n (\bar{t}_k)^N \mathbb{E} \left[\prod_{j=1}^{m+n} \phi(s_j) \prod_{k=1}^N e^{-in(\theta_k - \pi)} \right] = \prod_{k=1}^n (\bar{t}_k)^N \times$$

$$\frac{M_N(\bar{b} - n, b + n; \beta/2)}{M_N(\bar{b}, b; \beta/2)} {}_2F_1^{(\beta/2)}(-N, (2/\beta)(b + n); -N + 1 - (2/\beta)(\bar{b} - n + 1); s).$$

Notice

$$\frac{M_N(\bar{b} - n, b + n; \beta/2)}{M_N(\bar{b}, b; \beta/2)} = \prod_{j=0}^{n-1} \frac{\Gamma((2/\beta)(1 + b + j)) \Gamma(N + (2/\beta)(\bar{b} - j))}{\Gamma(N + (2/\beta)(1 + b + j)) \Gamma((2/\beta)(\bar{b} - j))},$$

by Proposition 13.1.7 and the formula (13.14) in [29, Chap. 13], simple manipulation of the gamma functions gives the desired result. \square

Next, we define a multivariate function which appears as a scaling limit near the spectrum singularity.

Definition 3.2. For $\alpha > 0$ and $\text{Re}\{b\} > -1/2$, we define a multivariate function

$$S_b^{(\alpha)}(x_1, \dots, x_{m+n}) = \gamma_{m,n}(b, 2/\alpha) \prod_{k=1}^{m+n} e^{-ix_k/2} \times$$

$${}_1F_1^{(\alpha)}((b + n)/\alpha; (\bar{b} + b + m + n)/\alpha; ix_1, \dots, ix_{m+n}) \quad (3.4)$$

where the constant

$$\gamma_{m,n}(b, 2/\alpha) = \prod_{j=0}^{m-1} \frac{\Gamma((1 + \bar{b} + j)/\alpha)}{\Gamma((1 + \bar{b} + b + n + j)/\alpha)} \prod_{j=0}^{n-1} \frac{\Gamma((1 + b + j)/\alpha)}{\Gamma((1 + \bar{b} + b + j)/\alpha)}. \quad (3.5)$$

Remark 3.3. Assume that $m = n$ and all x_j are real variables, then $S_b^{(\alpha)}(x)$ is a real-valued function, that is

$$S_b^{(\alpha)}(x) = \overline{S_b^{(\alpha)}(x)}.$$

This follows from $\gamma_{m,m}(b, 2/\alpha) = \gamma_{m,m}(\bar{b}, 2/\alpha)$ and the Kummer relation (see Proposition 3.2 [61])

$$\prod_{k=1}^n e^{x_k} {}_1F_1^{(\alpha)}(c - a; c; -x_1, \dots, -x_n) = {}_1F_1^{(\alpha)}(a; c; x_1, \dots, x_n).$$

In particular for $\alpha = 1$, application of (2.13) in Proposition 2.1 gives a determinant expression of the function $S_b^{(1)}(x)$, that is

$$S_b^{(1)}(x_1, \dots, x_{m+n}) = \gamma_{m,n}(b, 2) \prod_{j=0}^{m+n-1} \frac{j! (\bar{b} + b + 1)_j}{(b - m + 1)_j}$$

$$\times e^{-i \sum_{k=1}^{m+n} x_k/2} \frac{\det[(ix_j)^{k-1} {}_1f_1^{(k-1)}(ix_j)]}{\det[(ix_j)^{k-1}]}, \quad (3.6)$$

where

$${}_1f_1(z) := {}_1f_1(b - m + 1; \bar{b} + b + 1; z) = \sum_{k=0}^{\infty} \frac{(b - m + 1)_k}{(\bar{b} + b + 1)_k} \frac{z^k}{k!}. \quad (3.7)$$

Further, if $m = n = 1$, then

$$S_b^{(1)}(x_1, x_2) = \frac{\Gamma(\bar{b}+1)\Gamma(b+1)}{\Gamma(\bar{b}+b+1)\Gamma(\bar{b}+b+2)} e^{-i(x_1+x_2)/2} \frac{1}{x_2 - x_1} \\ \times (x_2 {}_1f_1(b; \bar{b}+b+1; ix_1) {}_1f_1(b+1; \bar{b}+b+2; ix_2) - (x_1 \leftrightarrow x_2)), \quad (3.8)$$

which is a new kernel and reduces to the sine kernel when $b = 0$.

Another special case is that when $m = n = 1$ and $b = 0$ for general α , $S_0^{(\alpha)}(x_1, x_2)$ can be expressed by the Bessel function $(x_2 - x_1)^{-\frac{1}{\alpha} + \frac{1}{2}} J_{\frac{1}{\alpha} - \frac{1}{2}}(x_2 - x_1)$, see Proposition 5.2 [17].

We now state the main results of this section as follows.

Theorem 3.4 (Limit at zero). *Let ρ be a nonzero constant. Introducing the scaled variables*

$$s_j = e^{i\frac{x_j}{\rho N}}, \quad j = 1, \dots, m \quad \text{and} \quad t_k = e^{i\frac{x_{m+k}}{\rho N}}, \quad k = 1, \dots, n$$

with real variables x_1, \dots, x_{m+n} , we have

$$\lim_{N \rightarrow \infty} N^{-2(mn+bm+\bar{b}n)/\beta} K_{b,N}(s; t) = \\ \prod_{j=1}^m e^{ix_j/(2\rho)} \prod_{k=1}^n e^{-ix_{m+k}/(2\rho)} S_b^{(\beta/2)}(x_1/\rho, \dots, ix_{m+n}/\rho) \quad (3.9)$$

where $K_{b,N}(s; t)$ is given in (1.12).

Note that when $b = 0$ ${}_0\mathcal{F}_0^{(\beta/2)}(x_1, \dots, x_{m+n}; 1^{(m)}, (-1)^{(n)})$ appears as a scaling limit in the non-trivial part $S_b^{(\beta/2)}(x)$ on the RHS of (3.9) for any integers m and n , after application of (2.17).

Proof. Notice the asymptotic formula of the gamma functions

$$\frac{\Gamma(N+b)}{\Gamma(N+a)} \sim N^{b-a} \quad \text{as } N \rightarrow \infty,$$

together with the following limit property (cf. (13.5), [29, Chap. 13])

$$\lim_{b' \rightarrow \infty} {}_2F_1^{(\alpha)}(a', b'; c'; s_1/b', \dots, s_m/b') = {}_1F_1^{(\alpha)}(a'; c'; s_1, \dots, s_m),$$

application of Proposition 3.1 gives the desired result. \square

When β is an even integer, the scaling limit of correlation functions immediately follows from Theorem 3.4.

Corollary 3.5 (Correlations at zero for even β). *Let ρ be a nonzero constant. Introducing the scaled variables*

$$r_j = \frac{y_j}{\rho N}, \quad j = 1, \dots, k$$

with real variables y_1, \dots, y_k in (1.13), we have

$$\lim_{N \rightarrow \infty} \left(\frac{1}{\rho N}\right)^k R_{b,N}^{(k)}\left(\frac{y_1}{\rho N}, \dots, \frac{y_k}{\rho N}\right) = R_{b,\rho}^{(k,\beta/2)}(y_1, \dots, y_k),$$

where

$$R_{b,\rho}^{(k,\beta/2)}(y_1, \dots, y_k) = c_k(b, \beta) (2\pi\rho)^{-k} e^{i\frac{b-\bar{b}}{2}k\pi} \prod_{j=1}^k (|y_j/\rho|^{\bar{b}+b} e^{i\frac{\beta}{2\rho}y_j}) |\Delta_k(y/\rho)|^\beta \times \\ {}_1F_1^{(\beta/2)}((2/\beta)b + k; (2/\beta)(\bar{b} + b) + 2k; -iy_1/\rho, \dots, -iy_k/\rho). \quad (3.10)$$

Here in the argument of ${}_1F_1^{(\beta/2)}$ each $-iy_j/\rho$ ($j = 1, \dots, k$) occurs β times, and

$$c_k(b, \beta) = (\beta/2)^{(\bar{b}+b)k + \beta k(k-1)/2} (\Gamma(1 + \beta/2))^k \times \\ \prod_{j=0}^{2k-1} \frac{1}{\Gamma(1 + \bar{b} + b + \beta j/2)} \prod_{j=0}^{k-1} \Gamma(1 + \bar{b} + \beta j/2) \Gamma(1 + b + \beta j/2). \quad (3.11)$$

Proof. Let $m = \beta k/2$. Introduce $s_j = e^{i\frac{x_j}{\rho N}}$, $j = 1, \dots, m$ and

$$x_{l+\beta(j-1)/2} = -y_j, \quad j = 1, \dots, k, l = 1, \dots, \beta/2,$$

by use of the formula (1.14) simple manipulation gives as $N \rightarrow \infty$

$$\left(\frac{1}{\rho N}\right)^k R_{b,N}^{(k)}\left(\frac{y_1}{\rho N}, \dots, \frac{y_k}{\rho N}\right) \sim \frac{M_N(\bar{b}, b; \beta/2)}{M_{k+N}(\bar{b}, b; \beta/2)} \rho^{-k} \times \\ N^{-(\bar{b}+b)k - \beta k(k-1)/2} e^{i\frac{b-\bar{b}}{2}k\pi} \prod_{j=1}^k |y_j/\rho|^{\bar{b}+b} |\Delta_k(y/\rho)|^\beta K_{b,N}(s; s). \quad (3.12)$$

Notice the notation M_N specified by (1.2) and one easily derives

$$\frac{M_N(\bar{b}, b; \beta/2)}{M_{k+N}(\bar{b}, b; \beta/2)} \sim (2\pi)^{-k} (\Gamma(1 + \beta/2))^k (\beta N/2)^{-\beta k/2}.$$

On the other hand, for Gauss's multiplication formulas of the gamma functions

$$\prod_{j=0}^{l-1} \Gamma(a + \frac{j}{l}) = l^{-la + \frac{1}{2}} (2\pi)^{\frac{l-1}{2}} \Gamma(la), \quad l \in \mathbb{N}, \quad (3.13)$$

let $l = \beta/2$ and we get (cf. (3.5))

$$\gamma_{m,m}(b, 4/\beta) = \frac{\prod_{p=0}^{k-1} \prod_{j=0}^{-1+\beta/2} \Gamma(\frac{1+\bar{b}}{\beta/2} + p + \frac{j}{\beta/2}) \Gamma(\frac{1+b}{\beta/2} + p + \frac{j}{\beta/2})}{\prod_{p=0}^{2k-1} \prod_{j=0}^{\frac{\beta}{2}-1} \Gamma(\frac{1+\bar{b}+b}{\beta/2} + p + \frac{j}{\beta/2})} \\ = \frac{\prod_{p=0}^{k-1} \Gamma(1 + \bar{b} + \beta p/2) \Gamma(1 + b + \beta p/2)}{\prod_{p=0}^{2k-1} \Gamma(1 + \bar{b} + b + \beta p/2)}. \quad (3.14)$$

Together with (3.12), the desired result immediately follows from Theorem 3.4. \square

We remark that in the circular β -ensemble, i.e., $b = 0$, Corollary 3.5 is actually identical to the bulk of the Gaussian, Laguerre and Jacobi ensembles [17], and has been proved by Forrester [27] (see also Proposition 13.2.3 of [29]). For $\beta = 1, 2, 4$ and fixed real b , the scaling limits for point correlation functions near the spectrum singularity (at zero) were proved by Forrester and Nagao [31], see also Chapter 7.2.6, [29] for the limiting kernel and relationship with the sine kernel in the $\beta = 2$ case. For $\beta = 2$ and fixed real b , the same universal correlation functions as in

Corollary 3.5 appear in a wide class of unitary ensembles with singularity at the origin, see [1, 2, 43, 49] and references therein.

Remark 3.6. Theorem 3.4 and Corollary 3.5 just involve the local limit at the angle $\theta = 0$ of the unit circle. However, when $b = 0$, the local limit still holds true near every angle of the unit circle because of the relation $K_{0,N}(e^{i\theta}s; e^{i\theta}t) = K_{0,N}(s; t)$ for any real θ . But for the general fixed b , our method may not work at the other angle than zero.

4. LIMITS FOR $b = \beta Nd/2$

We devote ourselves to the case of $b = \beta Nd/2$ and prove the local limits in the bulk and at the soft edge. In this section let $\alpha = \beta/2$, for convenience, α and β will be used in alternation.

4.1. General procedure. Using Proposition 3.1 with the same notation $s_{m+j} = (\bar{t}_j)^{-1}$ ($j = 1, \dots, n$), if $\text{Re}\{b\} > \max\{m, n\} - 1$, we then have the following integral representation due to Yan [61]

$$K_{b,N}(s; t) = C_N \prod_{k=1}^n (\bar{t}_k)^N \frac{1}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} \times \int_{[0,1]^{m+n}} {}_1\mathcal{F}_0^{(\alpha)}(-N; 1-s; y) \prod_{j=1}^{m+n} y_j^{\nu_1} (1-y_j)^{\nu_2} |\Delta_{m+n}(y)|^{2/\alpha} d^{m+n}y, \quad (4.1)$$

where $\nu_1 = -1 + (b - m + 1)/\alpha$, $\nu_2 = -1 + (\bar{b} - n + 1)/\alpha$ and

$$S_{m+n}(\nu_1, \nu_2; 1/\alpha) = \prod_{j=0}^{m+n-1} \frac{\Gamma(1 + (1+j)/\alpha) \Gamma(1 + \nu_1 + j/\alpha) \Gamma(1 + \nu_2 + j/\alpha)}{\Gamma(1 + 1/\alpha) \Gamma(2 + \nu_1 + \nu_2 + (m+n+j-1)/\alpha)}. \quad (4.2)$$

Introduce the scaled variables

$$s_j = e^{i\theta + i \frac{x_j}{\rho N}}, \quad j = 1, \dots, m+n,$$

where θ is a spectral parameter that allows us to select the part of the spectrum we are going to study, and ρ is to be determined. For convenience, let the new spectral parameter

$$u = \frac{1}{1 - e^{i\theta}}, \quad (4.3)$$

then $1 - s_j = \frac{1}{u} + w_j$ where $w_j = e^{i\theta} (1 - e^{i \frac{x_j}{\rho N}})$. We also assume $b = \alpha Nd$ where $\text{Re}\{d\} > 0$. Application of (2.15) gives

$$K_{b,N}(s; t) = \frac{C_N e^{-inN\theta}}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} \prod_{k=1}^n e^{-ix_{m+k}/\rho} \times \int_{[0,1]^{m+n}} \exp\left\{-N \sum_{j=1}^{m+n} p(y_j)\right\} |\Delta_{m+n}(y)|^{2/\alpha} \tilde{Q}(y) d^{m+n}y, \quad (4.4)$$

where

$$p(y_j) = -d \log y_j - \bar{d} \log(1 - y_j) - \log(1 - u^{-1}y_j), \quad (4.5)$$

and

$$\tilde{Q}(y) = \prod_{j=1}^{m+n} y_j^{\frac{1-m}{\alpha}-1} (1-y_j)^{\frac{1-n}{\alpha}-1} {}_1\mathcal{F}_0^{(\alpha)}(-N; w; \frac{y}{1-u^{-1}y}). \quad (4.6)$$

Notice as $N \rightarrow \infty$ one has $w_j \sim -ie^{i\theta} x_j/(\rho N)$, and

$$\tilde{Q}(y) = \prod_{j=1}^{m+n} y_j^{\frac{1-m}{\alpha}-1} (1-y_j)^{\frac{1-n}{\alpha}-1} \left({}_0\mathcal{F}_0^{(\alpha)}(x/\rho; \frac{ie^{i\theta}y}{1-u^{-1}y}) + O(\frac{1}{N}) \right),$$

which shows

$$K_{b,N}(s; t) \sim \frac{C_N e^{-inN\theta}}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} \prod_{k=1}^n e^{-ix_{m+k}/\rho} I_N. \quad (4.7)$$

Here

$$I_N = \int_{[0,1]^{m+n}} \exp\{-N \sum_{j=1}^{m+n} p(y_j)\} |\Delta_{m+n}(y)|^{2/\alpha} Q(y) d^{m+n}y \quad (4.8)$$

and

$$Q(y) = \prod_{j=1}^{m+n} y_j^{\frac{1-m}{\alpha}-1} (1-y_j)^{\frac{1-n}{\alpha}-1} {}_0\mathcal{F}_0^{(\alpha)}(x/\rho; \frac{ie^{i\theta}y}{1-u^{-1}y}). \quad (4.9)$$

From now on we suppose that d is a positive real number, we then have

$$p(y_j) = -d \log y_j - d \log(1-y_j) - \log(1-u^{-1}y_j). \quad (4.10)$$

However, our method may be applicable to the general case of $\text{Re}\{d\} > 0$, except that some much more complicated computation must be done.

We will make use of Corollaries (3.11) and (3.12) of [17], which are based on the steepest descent method for Selberg-type integrals, to asymptotically evaluate I_N as $N \rightarrow \infty$. In our case, since

$$p'(x) = -\frac{d}{x} + \frac{d}{1-x} + \frac{1}{u-x},$$

there are at most two saddle points x_{\pm} , which satisfy

$$x_{\pm} = \frac{1}{2} + \frac{1}{2(1+\sin \frac{\theta_d}{2}) \sin \frac{\theta}{2}} \left(\pm \sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_d}{2}} + i \sin \frac{\theta_d}{2} \cos \frac{\theta}{2} \right). \quad (4.11)$$

Recall that $\theta_d \in [0, \pi)$ and is defined by

$$\sin \frac{\theta_d}{2} = \frac{d}{1+d}. \quad (4.12)$$

Also let $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be such that

$$\cos \varphi = \frac{\sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_d}{2}}}{\cos \frac{\theta_d}{2} \sin \frac{\theta}{2}}, \quad \sin \varphi = \frac{\sin \frac{\theta_d}{2} \cos \frac{\theta}{2}}{\cos \frac{\theta_d}{2} \sin \frac{\theta}{2}}, \quad (4.13)$$

then we have

$$x_+ = \frac{1}{2} + \frac{\cos \frac{\theta_d}{2}}{2(1+\sin \frac{\theta_d}{2})} e^{i\varphi}, \quad x_- = \frac{1}{2} + \frac{\cos \frac{\theta_d}{2}}{2(1+\sin \frac{\theta_d}{2})} e^{i(\pi-\varphi)}. \quad (4.14)$$

The nature of the saddle points depends on the value of θ , and we distinguish two cases:

- (1) Two complex saddle points of degree one x_+, x_- given in (4.14) when $\theta \in (\theta_d, 2\pi - \theta_d)$, i.e. $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
- (2) One saddle point of degree two

$$x_0 = \frac{1}{2} + \frac{i \cos \frac{\theta_d}{2}}{2(1 + \sin \frac{\theta_d}{2})} \text{ for } \theta = \theta_d \quad (4.15)$$

while

$$x_0 = \frac{1}{2} - \frac{i \cos \frac{\theta_d}{2}}{2(1 + \sin \frac{\theta_d}{2})} \text{ for } \theta = 2\pi - \theta_d. \quad (4.16)$$

In the following subsections, we proceed to compute the leading asymptotic terms for products of characteristic polynomials by use of Corollaries 3.11 and 3.12 in [17], respectively in the bulk and at the soft edge of the spectrum.

4.2. Bulk limits.

Theorem 4.1 (Bulk limit). *Let*

$$\rho = \frac{1}{2\pi} \frac{\sqrt{\sin^2(\theta/2) - \sin^2(\theta_d/2)}}{(1 - \sin(\theta_d/2)) \sin(\theta/2)}, \quad \theta_d < \theta < 2\pi - \theta_d,$$

and introduce the scaled variables

$$s_j = e^{i\theta + i\frac{x_j}{\rho N}}, \quad j = 1, \dots, m \quad \text{and} \quad t_k = e^{i\theta + i\frac{x_{m+k}}{\rho N}}, \quad k = 1, \dots, n$$

with real variables x_1, \dots, x_{m+n} . Assume that $m + n = 2l$ and $b = \beta N d/2$ with $d > 0$, then as $N \rightarrow \infty$ we have

$$\begin{aligned} \frac{K_{b,N}(s; t)}{\Psi_{N,m,n}} &\sim \exp \left\{ \frac{1}{2\rho} \left[(i - d \cot \frac{\theta}{2}) \sum_{j=1}^m x_j - (i + d \cot \frac{\theta}{2}) \sum_{j=1}^n x_{m+j} \right] \right\} \\ &\quad \times \gamma_l(4/\beta) e^{-i\pi \sum_{k=1}^{2l} x_k} {}_1F_1^{(\beta/2)}(2l/\beta; 4l/\beta; 2i\pi x) \end{aligned} \quad (4.17)$$

where $K_{b,N}(s; t)$ is defined in (1.12), $\gamma_l(4/\beta) := \gamma_{l,l}(0, 4/\beta)$ is given in (3.5) and

$$\begin{aligned} \Psi_{N,m,n} &= (2\pi\rho)^{\frac{2l(l+1)}{\beta} - l} e^{i\frac{m-n}{2}N\theta + i\frac{m-n}{\beta}l(\theta - \pi)} (2 \sin \frac{\theta}{2})^{-2ldN} N^{\frac{2l^2}{\beta}} d^{\frac{(m-n)^2}{2\beta}} \\ &\quad \times (1+d)^{-2l(1+d)N - \frac{m(m+1)+n(n+1)}{\beta} + l} (1+2d)^{l(1+2d)N + \frac{l(l+1)}{\beta} - \frac{l}{2}}. \end{aligned} \quad (4.18)$$

Proof. We now continue the foregoing analysis in Section 4.1. According to Corollary 3.12 in [17], we will perform some simple calculations as follows.

First, together with (4.10) and (4.14) simple manipulation shows

$$\begin{aligned} p(x_{\pm}) &= \mp i \left(\frac{\pi}{2} + d\varphi \right) + (1+d) \log \frac{\sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_d}{2}} \pm i \cos \frac{\theta}{2}}{\cos \frac{\theta_d}{2}} \\ &\quad - i \frac{\theta}{2} + \log \frac{1 + \sin \frac{\theta_d}{2}}{\cos \frac{\theta_d}{2}} + d \log 2 + d \log \left(\frac{1 + \sin \frac{\theta_d}{2}}{\sin \frac{\theta_d}{2}} \sin \frac{\theta}{2} \right), \end{aligned} \quad (4.19)$$

and

$$p''(x_{\pm}) = \frac{4\sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_d}{2}}}{\sin \frac{\theta_d}{2} (1 - \sin \frac{\theta_d}{2})^2} \cos \frac{\theta_d}{2} \sin \frac{\theta}{2} (1 + \sin \frac{\theta_d}{2}) \times \\ \left\{ \pm i \frac{\cos \frac{\theta}{2} (2 \sin^2 \frac{\theta}{2} + \sin \frac{\theta_d}{2} - \sin^2 \frac{\theta_d}{2})}{\cos \frac{\theta_d}{2} \sin \frac{\theta}{2} (1 + \sin \frac{\theta_d}{2})} + \frac{2 \sin^2 \frac{\theta}{2} + \sin \frac{\theta_d}{2} - 1}{\cos \frac{\theta_d}{2} \sin \frac{\theta}{2} (1 + \sin \frac{\theta_d}{2})} \sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_d}{2}} \right\}. \quad (4.20)$$

Secondly, notice the formulas (4.8)–(4.11) and by Corollary 3.12 of [17] we get for even $m + n = 2l$

$$I_N \sim \binom{2l}{l} (\Gamma_{2/\alpha, l})^2 \frac{(x_+ - x_-)^{2l^2/\alpha}}{(\sqrt{p''(x_+)p''(x_-)})^{l+l(l-1)/\alpha}} \frac{e^{-lN(p(x_+)+p(x_-))}}{N^{l+l(l-1)/\alpha}} Q(x_+^{(l)}, x_-^{(l)}) \quad (4.21)$$

where a special case of Selberg integrals [29]

$$\int_{\mathbb{R}^n} \prod_{k=1}^l e^{-x_k^2/2} \prod_{1 \leq i < j \leq l} |x_j - x_i|^\beta d^l x = \Gamma_{\beta, l} \quad (4.22)$$

reads off

$$\Gamma_{\beta, l} = (2\pi)^{l/2} \prod_{j=1}^l \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)}. \quad (4.23)$$

Since a direct computation gives

$$\frac{ie^{i\theta}x_{\pm}}{1 - u^{-1}x_{\pm}} = \frac{1}{2 \sin \frac{\theta}{2}} \left(e^{i\frac{\theta}{2}} - \frac{\cos \frac{\theta}{2}}{1 - \sin \frac{\theta_d}{2}} \right) \pm \frac{i\sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_d}{2}}}{2 \sin \frac{\theta}{2} (1 - \sin \frac{\theta_d}{2})}, \quad (4.24)$$

substitution of (4.11), (4.19), (4.20) into the RHS of (4.21) and application of the translate property (2.16), after minor manipulation we get

$$I_N \sim \binom{2l}{l} (\Gamma_{2/\alpha, l})^2 N^{-\frac{l(l-1)}{\alpha} - l} \times \\ (2\pi\rho)^{\frac{l(l+1)}{\alpha} - l} e^{i\frac{l}{2\alpha}(m-n)(\theta-\pi)} \left(\frac{1 - \sin \frac{\theta_d}{2}}{1 + \sin \frac{\theta_d}{2}} \right)^{\frac{2l^2}{\alpha}} \left(\tan \frac{\theta_d}{2} \right)^{-\frac{l(l-1)}{\alpha} - l} \times \\ \exp \left\{ -lN \left[-i\theta + 2 \log \frac{1 + \sin \frac{\theta_d}{2}}{\cos \frac{\theta_d}{2}} + 2d \log 2 + 2d \log \frac{(1 + \sin \frac{\theta_d}{2}) \sin \frac{\theta}{2}}{\sin \frac{\theta_d}{2}} \right] \right\} \times \\ \exp \left\{ \frac{1}{2\rho} \left[(i - d \cot \frac{\theta}{2}) \sum_{j=1}^m x_j - (i + d \cot \frac{\theta}{2}) \sum_{j=1}^n x_{m+j} \right] \right\} {}_0\mathcal{F}_0^{(\alpha)}(i\pi x; 1^{(l)}, (-1)^{(l)}).$$

On the other hand, use of the asymptotic formula for the gamma functions gives as $N \rightarrow \infty$

$$\frac{C_N}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} \sim (2\pi)^{-\frac{m+n}{2}} (1 + 2d)^{(1+2d)(m+n)N + \frac{(m+n)(m+n+1)}{\beta} - \frac{m+n}{2}} \\ \times (1 + d)^{-(1+d)(m+n)N - \frac{m(m+1)+n(n+1)}{\beta} + \frac{m+n}{2}} d^{-d(m+n)N + \frac{(m-1)m+(n-1)n}{\beta} + \frac{m+n}{2}} \\ \times N^{\frac{(m+n)(m+n-1)}{\beta} + \frac{m+n}{2}} \prod_{j=0}^{m+n-1} \frac{\Gamma(1 + 2/\beta)}{\Gamma(1 + 2(1+j)/\beta)}. \quad (4.25)$$

Finally, combine the asymptotics of I_N with the above and also notice (4.7), (4.12), (4.23) and (2.17), we thus complete the proof. \square

When β is an even integer, the scaling limit of correlation functions immediately follows from Theorem 4.1.

Corollary 4.2 (Bulk correlations for even β). *Let*

$$\rho = \frac{1}{2\pi} \frac{\sqrt{\sin^2(\theta/2) - \sin^2(\theta_d/2)}}{(1 - \sin(\theta_d/2)) \sin(\theta/2)}, \quad \theta_d < \theta < 2\pi - \theta_d,$$

and introduce the scaled variables

$$r_j = \theta + \frac{y_j}{\rho N}, \quad j = 1, \dots, k$$

with real variables y_1, \dots, y_k in (1.13). Assume that β is even and $b = \beta N d/2$ with $d > 0$, then

$$\begin{aligned} \lim_{N \rightarrow \infty} \left(\frac{1}{\rho N} \right)^k R_{b,N}^{(k)} \left(\theta + \frac{y_1}{\rho N}, \dots, \theta + \frac{y_k}{\rho N} \right) &= c_k(0, \beta) |\Delta_k(2\pi y)|^\beta \\ &\times e^{-i\beta\pi \sum_{j=1}^k y_j} {}_1F_1^{(\beta/2)}(k; 2k; 2i\pi y_1, \dots, 2i\pi y_1, \dots, 2i\pi y_k, \dots, 2i\pi y_k) \end{aligned} \quad (4.26)$$

where in the argument of ${}_1F_1^{(\beta/2)}$ each $2i\pi y_j$ ($j = 1, \dots, k$) occurs β times, and $c_k(0, \beta)$ is defined in (3.11).

Proof. Let $m = \beta k/2$. Introduce $s_j = e^{i(2\pi - \theta) + i\frac{x_j}{\rho N}}$, $j = 1, \dots, m$ and

$$x_{l+\beta(j-1)/2} = -y_j, \quad j = 1, \dots, k, l = 1, \dots, \beta/2,$$

by use of the formula (1.15) as $N \rightarrow \infty$ we get

$$\begin{aligned} \left(\frac{1}{\rho N} \right)^k R_{b,N}^{(k)} \left(\theta + \frac{y_1}{\rho N}, \dots, \theta + \frac{y_k}{\rho N} \right) &\sim \frac{M_N(b, b; \beta/2)}{M_{k+N}(b, b; \beta/2)} \rho^{-k - \beta k(k-1)/2} \times \\ &N^{-\beta k(k-1)/2} (2 \sin \frac{\theta}{2})^{\beta d k N} \exp \left\{ \frac{\beta d}{2\rho} \cot \frac{\theta}{2} \sum_{j=1}^k y_j \right\} |\Delta_k(y)|^\beta K_{b,N}(s; s). \end{aligned} \quad (4.27)$$

Substitute θ by $2\pi - \theta$ in Theorem 4.1, and notice the following asymptotics (cf. the formula (1.2))

$$\begin{aligned} \frac{M_N(b, b; \beta/2)}{M_{k+N}(b, b; \beta/2)} &\sim (2\pi)^{-k} (\Gamma(1 + \beta/2))^k (\beta N/2)^{-\beta k/2} \times \\ &(1+d)^{\beta k(k-1)/2 + \beta k(1+d)N+k} (1+2d)^{-\beta k(k-1)/4 - \beta k(1+2d)N/2 - k/2}, \end{aligned} \quad (4.28)$$

we have

$$\begin{aligned} \left(\frac{1}{\rho N} \right)^k R_{b,N}^{(k)} \left(\theta + \frac{y_1}{\rho N}, \dots, \theta + \frac{y_k}{\rho N} \right) &\sim (2\pi)^{\beta k(k-1)/2} (\beta/2)^{-\beta k/2} \times \\ &(\Gamma(1 + \beta/2))^k \gamma_m(4/\beta) |\Delta_k(y)|^\beta {}_0\mathcal{F}_0^{(\alpha)}(i\pi x; 1^{(m)}, (-1)^{(m)}). \end{aligned} \quad (4.29)$$

Finally, use the Gauss's multiplication formulas of the gamma functions (cf. the equation (3.13)) for $\gamma_m(4/\beta)$ and we thus complete the proof. \square

4.3. Soft edge limits. The asymptotic results at the soft edge will be written in terms of multivariate Airy functions of x_1, \dots, x_n [15, 17] defined by

$$\text{Ai}_n^{(\alpha)}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \sum_{j=1}^n w_j^3/3} |\Delta_n(w)|^{2/\alpha} {}_0\mathcal{F}_0^{(\alpha)}(x; iw) d^n w. \quad (4.30)$$

Note that, by convention, the notation \pm refers to the sign of the involved quantities corresponding to the spectral edge $\theta = \theta_d$ and $\theta = 2\pi - \theta_d$, respectively.

Theorem 4.3 (Soft edge limit). *Let*

$$\rho = \pm(1 - \sin \frac{\theta_d}{2})^{-2/3} (-\cot \frac{\theta_d}{2})^{1/3} (4N)^{-1/3},$$

and introduce the scaled variables

$$s_j = e^{i\theta + i \frac{x_j}{\rho N}}, \quad j = 1, \dots, m \quad \text{and} \quad t_k = e^{i\theta + i \frac{x_{m+k}}{\rho N}}, \quad k = 1, \dots, n$$

where $\theta = \theta_d$ or $\theta = 2\pi - \theta_d$. Assume that $b = \beta N d/2$ with $d > 0$, then as $N \rightarrow \infty$ we have

$$\begin{aligned} \frac{K_{b,N}(s; t)}{\Phi_{N,m,n}} &\sim \exp \left\{ \frac{1}{2\rho} \left[(i \pm \frac{-\cos \frac{\theta_d}{2}}{1 - \sin \frac{\theta_d}{2}}) \sum_{j=1}^m x_j - (i \pm \frac{\cos \frac{\theta_d}{2}}{1 - \sin \frac{\theta_d}{2}}) \sum_{j=1}^n x_{m+j} \right] \right\} \\ &\times (2\pi)^{m+n} (\Gamma_{4/\beta, m+n})^{-1} \text{Ai}_{m+n}^{(\beta/2)}(x_1, \dots, x_{m+n}). \end{aligned} \quad (4.31)$$

Here $K_{b,N}(s; t)$ is defined in (1.12), $\Gamma_{4/\beta, m+n}$ is given in (4.23) and

$$\begin{aligned} \Phi_{N,m,n} &= 2^{-(m+n)dN + \frac{1}{3}(m+n) - \frac{1}{3\beta}(m+n)(m+n+2)} d^{-(m+n)dN + \frac{1}{3\beta}(m^2+n^2-4mn-m-n)} \\ &\times (1+d)^{-(m+n)N + \frac{1}{6}(m+n) - \frac{1}{3\beta}(2m^2+2n^2-2mn+m+n)} \\ &\times (1+2d)^{-\frac{1}{2}(m+n)(1+2d)N - \frac{1}{3}(m+n) + \frac{1}{3\beta}(m+n)(m+n+2)} \\ &\times \left(\frac{\sqrt{1+2d} - i}{\sqrt{1+2d} + i} \right)^{\pm(m-n)(m+n)/\beta} e^{\pm i(m-n)N\theta_d/2} N^{\frac{1}{6}(m+n) + \frac{1}{3\beta}(m+n)(m+n-1)}. \end{aligned} \quad (4.32)$$

Proof. As in the bulk case we now continue the foregoing analysis in Section 4.1. But this time we will make use of Corollary 3.11 in [17], which deals with one saddle point of higher order. Since the two cases of $\theta = \theta_d$ and $\theta = 2\pi - \theta_d$ are similar, we only consider the former.

First, at $\theta = \theta_d$ simple manipulation from (4.10) and (4.15) shows

$$p(x_0) = -i \frac{\theta_d}{2} + \log \frac{1 + \sin \frac{\theta_d}{2}}{\cos \frac{\theta_d}{2}} + d \log 2 + d \log \left(1 + \sin \frac{\theta_d}{2} \right), \quad (4.33)$$

and

$$p'''(x_0) = \frac{8i(1 + \sin \frac{\theta_d}{2})^3}{(1 - \sin \frac{\theta_d}{2}) \cot \frac{\theta_d}{2}}. \quad (4.34)$$

On the other hand,

$$\frac{ie^{i\theta} x_0}{1 - u^{-1} x_0} = \frac{i}{2} - \frac{\cos \frac{\theta_d}{2}}{2(1 - \sin \frac{\theta_d}{2})}, \quad (4.35)$$

thus by (2.16) we can rewrite (4.8) as

$$I_N = \exp \left\{ \frac{1}{2\rho} \left(i - \frac{\cos \frac{\theta_d}{2}}{1 - \sin \frac{\theta_d}{2}} \right) \sum_{j=1}^{m+n} x_j \right\} \\ \times \int_{[0,1]^{m+n}} \exp \left\{ -N \sum_{j=1}^{m+n} p(y_j) \right\} |\Delta_{m+n}(y)|^{2/\alpha} q(y) g(y) d^{m+n} y \quad (4.36)$$

where

$$q(y) = \prod_{j=1}^{m+n} y_j^{\frac{1-m}{\alpha}-1} (1-y_j)^{\frac{1-n}{\alpha}-1}, \quad (4.37)$$

and

$$g(y) = {}_0\mathcal{F}_0^{(\alpha)}(x/\rho; \frac{ie^{i\theta}y}{1-u^{-1}y} - \frac{ie^{i\theta}x_0}{1-u^{-1}x_0}). \quad (4.38)$$

Secondly, notice the expansion at x_0

$$\frac{ie^{i\theta}y_j}{1-u^{-1}y_j} - \frac{ie^{i\theta}x_0}{1-u^{-1}x_0} = \frac{i(1+\sin \frac{\theta_d}{2})}{1-\sin \frac{\theta_d}{2}}(y_j - x_0) + \dots,$$

let

$$\rho = (1 - \sin \frac{\theta_d}{2})^{-2/3} (-\cot \frac{\theta_d}{2})^{1/3} (4N)^{-1/3},$$

then by Corollary 3.11 of [17] (the situation is a little different, but the proof there can be easily modified to be available for the form of (4.38)) we get

$$I_N \sim \exp \left\{ \frac{1}{2\rho} \left(i - \frac{\cos \frac{\theta_d}{2}}{1 - \sin \frac{\theta_d}{2}} \right) \sum_{j=1}^{m+n} x_j \right\} \frac{e^{-(m+n)Np(x_0)}}{N^{\frac{(m+n)(m+n-1)}{3\alpha} + \frac{m+n}{3}}} q(x_0, \dots, x_0) \\ \times \int_{\mathbb{R}^{m+n}} e^{-\frac{p'''(x_0)}{6} \sum_{j=1}^n w_j^3} |\Delta_n(w)|^{2/\alpha} {}_0\mathcal{F}_0^{(\alpha)} \left(\frac{x}{\rho N^{1/3}}; \frac{i(1+\sin \frac{\theta_d}{2})}{1-\sin \frac{\theta_d}{2}} w \right) d^{m+n} w.$$

Substitution of (4.33) and (4.34), it could simplify to

$$I_N \sim N^{-\frac{(m+n)(m+n-1)}{3\alpha} - \frac{m+n}{3}} \exp \left\{ \frac{1}{2\rho} \left(i - \frac{\cos \frac{\theta_d}{2}}{1 - \sin \frac{\theta_d}{2}} \right) \sum_{j=1}^{m+n} x_j \right\} \\ \times \exp \left\{ -(m+n)N \left(-i \frac{\theta_d}{2} + \log \frac{1 + \sin \frac{\theta_d}{2}}{\cos \frac{\theta_d}{2}} + d \log 2 + d \log(1 + \sin \frac{\theta_d}{2}) \right) \right\} \\ \times \left(\frac{4(1 + \sin \frac{\theta_d}{2})^3}{(1 - \sin \frac{\theta_d}{2}) \cot \frac{\theta_d}{2}} \right)^{-\frac{(m+n)(m+n-1)}{3\alpha} - \frac{m+n}{3}} \left(\frac{1}{2} + \frac{i \cos \frac{\theta_d}{2}}{2(1 + \sin \frac{\theta_d}{2})} \right)^{(m+n)(\frac{1-m}{\alpha}-1)} \\ \times \left(\frac{1}{2} - \frac{i \cos \frac{\theta_d}{2}}{2(1 + \sin \frac{\theta_d}{2})} \right)^{(m+n)(\frac{1-n}{\alpha}-1)} (2\pi)^{m+n} \text{Ai}_{m+n}^{(\alpha)}(x_1, \dots, x_{m+n}). \quad (4.39)$$

Finally, combine (4.7), (4.12), (4.25) and (4.39), we thus complete the proof after careful calculation. \square

When β is an even integer, the scaling edge limit of correlation functions immediately follows from Theorem 4.3.

Corollary 4.4 (Soft edge correlations for even β). *Let*

$$\rho = \pm(1 - \sin \frac{\theta_d}{2})^{-2/3} (-\cot \frac{\theta_d}{2})^{1/3} (4N)^{-1/3},$$

and introduce the scaled variables

$$r_j = \theta + \frac{y_j}{\rho N}, \quad j = 1, \dots, k$$

in (1.13). Assume that β is even and $b = \beta N d/2$ with $d > 0$, then for $\theta = \theta_d$ or $\theta = 2\pi - \theta_d$ we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \left(\frac{1}{|\rho|N} \right)^k R_{b,N}^{(k)} \left(\theta + \frac{y_1}{\rho N}, \dots, \theta + \frac{y_k}{\rho N} \right) \\ = a_k(\beta) |\Delta_k(y)|^\beta \text{Ai}_{\beta k}^{(\beta/2)}(y_1, \dots, y_1, \dots, y_k, \dots, y_k) \end{aligned} \quad (4.40)$$

where in the argument of $\text{Ai}_{\beta k}^{(\beta/2)}$ each y_j ($j = 1, \dots, k$) occurs β times, and

$$a_k(\beta) = (\beta/2)^{(\beta k+1)k} (\Gamma(1 + \beta/2))^k \prod_{j=1}^{2k} \frac{(\Gamma(1 + 2/\beta))^{\beta/2}}{\Gamma(1 + \beta j/2)}. \quad (4.41)$$

Proof. Let $m = \beta k/2$. Introduce $s_j = e^{i(2\pi - \theta) + i \frac{x_j}{\rho N}}$, $j = 1, \dots, m$ and

$$x_{l+\beta(j-1)/2} = -y_j, \quad j = 1, \dots, k, l = 1, \dots, \beta/2,$$

by use of the formula (1.15) as $N \rightarrow \infty$ we get

$$\begin{aligned} \left(\frac{1}{|\rho|N} \right)^k R_{b,N}^{(k)} \left(\theta + \frac{y_1}{\rho N}, \dots, \theta + \frac{y_k}{\rho N} \right) \sim \frac{M_N(b, b; \beta/2)}{M_{k+N}(b, b; \beta/2)} |\rho|^{-k - \beta k(k-1)/2} \times \\ N^{-\beta k(k-1)/2} (2 \sin \frac{\theta}{2})^{\beta d k N} \exp \left\{ \frac{\beta d}{2\rho} \cot \frac{\theta}{2} \sum_{j=1}^k y_j \right\} |\Delta_k(y)|^\beta K_{b,N}(s; s). \end{aligned} \quad (4.42)$$

Substitute θ by $2\pi - \theta$ in Theorem 4.3, and notice the asymptotics of (4.28), application of Theorem 4.3 after some cumbersome calculations and clever cancellation gives

$$\begin{aligned} \left(\frac{1}{\rho N} \right)^k R_{b,N}^{(k)} \left(\theta + \frac{y_1}{\rho N}, \dots, \theta + \frac{y_k}{\rho N} \right) \sim (2\pi)^{m-k} (\beta/2)^{-\beta k/2} (\Gamma(1 + \beta/2))^k \\ \times \prod_{j=1}^{2m} \frac{(\Gamma(1 + 2/\beta))^{\beta/2}}{\Gamma(1 + 2j/\beta)} |\Delta_k(y)|^\beta \text{Ai}_{\beta k}^{(\beta/2)}(y_1, \dots, y_1, \dots, y_k, \dots, y_k). \end{aligned} \quad (4.43)$$

Finally, use the Gauss's multiplication formulas of the gamma functions (cf. the equation (3.13)) for the product

$$\prod_{j=1}^{2m} \frac{(\Gamma(1 + 2/\beta))^{\beta/2}}{\Gamma(1 + 2j/\beta)}$$

and we thus complete the proof. \square

5. SPECTRUM SINGULARITY-TO-SOFT EDGE TRANSITION

In this section we describe the transition from the limiting multivariate function near the spectrum singularity to the multivariate Airy function at the soft edge as b goes to infinity. Similar results also hold for correlation functions when β is even.

We first point out some obvious properties of the function $S_b^{(\alpha)}(x)$ defined by (3.4) which describes the scaling limit near the spectrum singularity. With $b = 0$ the circular Jacobi ensemble corresponds to Dyson's circular ensemble, thus we have proved that $S_0^{(\alpha)}(x)$ is actually the bulk limit for the latter. Particularly when $m = n$ it is locally identical to the bulk limit of the Hermite (Gaussian), Laguerre (Chiral) and Jacobi ensembles proved in [17], see also Theorem 4.1. Another relationship between (3.4) and the bulk limit follows from the more general identity

$$S_b^{(\alpha)}(x_1, \dots, x_{m+n}, 0, 0) = \frac{\Gamma(\frac{1+\bar{b}}{\alpha}) \Gamma(\frac{1+b}{\alpha})}{\Gamma(\frac{1+\bar{b}+b}{\alpha}) \Gamma(\frac{2+\bar{b}+b}{\alpha})} S_{b+1}^{(\alpha)}(x_1, \dots, x_{m+n}), \quad (5.1)$$

which can be easily derived from the fact

$$P_\kappa^{(\alpha)}(x_1, \dots, x_n, 0) = P_\kappa^{(\alpha)}(x_1, \dots, x_n).$$

We now state the transition results as follows.

Theorem 5.1. *Assume that $b \in \mathbb{R}$. Introduce the scaled variables*

$$x_j = (2b/\alpha) - (4b/\alpha)^{1/3} \eta_j, \quad j = 1, \dots, m+n,$$

as $b \rightarrow \infty$ we have

$$\begin{aligned} S_b^{(\alpha)}(x) &\sim (-1)^{m+n} 2^{-\frac{b}{\alpha}(m+n) + \frac{n-2m-1}{3\alpha}(m+n) + \frac{m+n}{3}} (1-i)^{\frac{m^2-n^2}{\alpha}} e^{\frac{1}{2}(\frac{4b}{\alpha})^{1/3} \sum_{j=1}^{m+n} \eta_j} \\ &(b/\alpha)^{\frac{m^2+n^2-4mn-m-n}{6\alpha} - \frac{b(m+n)}{\alpha} + \frac{m+n}{6}} (2\pi)^{m+n} (\Gamma_{2/\alpha, m+n})^{-1} \text{Ai}_{m+n}^{(\alpha)}(\eta) \end{aligned} \quad (5.2)$$

where $\Gamma_{2/\alpha, m+n}$ is given in (4.23).

Proof. For $\text{Re}\{b\} > \max\{m, n\} - 1$, by the integral representation due to Yan [61] we have

$$\begin{aligned} S_b^{(\alpha)}(x) &= \frac{\gamma_{m,n}(b, 2/\alpha)}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} e^{-\frac{1}{2} \sum_{j=1}^{m+n} x_j} \times \\ &\int_{[0,1]^{m+n}} {}_0\mathcal{F}_0^{(\alpha)}(ix; y) \prod_{j=1}^{m+n} y_j^{\nu_1} (1-y_j)^{\nu_2} |\Delta_{m+n}(y)|^{2/\alpha} d^{m+n}y, \end{aligned} \quad (5.3)$$

where $\nu_1 = -1 + (b-m+1)/\alpha$, $\nu_2 = -1 + (b-n+1)/\alpha$ and $S_{m+n}(\nu_1, \nu_2; 1/\alpha)$ is defined as in (4.2).

With $x_j = (2b/\alpha) - (4b/\alpha)^{1/3} \eta_j$ application of (2.16) gives

$$\begin{aligned} S_b^{(\alpha)}(x) &= \frac{\gamma_{m,n}(b, 2/\alpha)}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} e^{-\frac{1}{2} \sum_{j=1}^{m+n} x_j} \int_{[0,1]^{m+n}} \prod_{j=1}^{m+n} y_j^{\frac{1-m}{\alpha}-1} (1-y_j)^{\frac{1-n}{\alpha}-1} \\ &\times e^{-\frac{b}{\alpha} \sum_{j=1}^{m+n} p(y_j)} {}_0\mathcal{F}_0^{(\alpha)}(-i(\frac{4b}{\alpha})^{1/3} \eta; y - \frac{1+i}{2}) |\Delta_{m+n}(y)|^{2/\alpha} d^{m+n}y, \end{aligned} \quad (5.4)$$

where

$$p(z) = -2i(z - \frac{1+i}{2}) - \log z - \log(1-z). \quad (5.5)$$

We thus derive from

$$p'(z) = -2i - \frac{1}{z} + \frac{1}{1-z} = 0 \quad (5.6)$$

that the saddle point $z_0 = \frac{1+i}{2}$. Obviously, $p''(z_0) = 0$, $p'''(z_0) = 8i$ and $p(z_0) = \log 2$.

Let's deform the interval $[0, 1]$ to the contour \mathcal{C} consisting of three straight line paths from $(0, 0)$ to $(0, i/2)$, then from $(0, i/2)$ to $(1, i/2)$ and then from $(1, i/2)$ to $(1, 0)$. It is easy to check that $\text{Re}\{p(z)\}$ attains its global minimum at $z = z_0$. Hence according to Corollary 3.11 [17] as $b \rightarrow \infty$ we have

$$\begin{aligned} S_b^{(\alpha)}(x) &\sim \frac{\gamma_{m,n}(b, 2/\alpha)}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} e^{-\frac{1}{2} \sum_{j=1}^{m+n} x_j} e^{-\frac{b}{\alpha}(m+n)p(z_0)} (z_0^{\frac{1-m}{\alpha}-1} (1-z_0)^{\frac{1-n}{\alpha}-1})^{m+n} \\ &\times \int_{\mathbb{R}^{m+n}} e^{-\frac{b}{\alpha} \frac{p'''(z_0)}{6} \sum_{j=1}^{m+n} w_j^3} \mathcal{F}_0^{(\alpha)}(-i(\frac{4b}{\alpha})^{1/3} \eta; w) |\Delta_{m+n}(w)|^{2/\alpha} d^{m+n} w \\ &= \frac{\gamma_{m,n}(b, 2/\alpha)}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} e^{-\frac{1}{2} \sum_{j=1}^{m+n} x_j} (-1)^{m+n} 2^{-\frac{b}{\alpha}(m+n) + (\frac{n-1}{\alpha}+1)(m+n)} (1-i)^{\frac{m^2-n^2}{\alpha}} \\ &\times (4b/\alpha)^{-\frac{(m+n)(m+n-1)}{3\alpha} - \frac{m+n}{3}} (2\pi)^{m+n} \text{Ai}_{m+n}^{(\alpha)}(\eta). \end{aligned} \quad (5.7)$$

Using the Stirling formula we can derive

$$\begin{aligned} \frac{\gamma_{m,n}(b, 2/\alpha)}{S_{m+n}(\nu_1, \nu_2; 1/\alpha)} &\sim (2\pi)^{-\frac{m+n}{2}} \prod_{j=1}^{m+n} \frac{\Gamma(1 + \frac{1}{\alpha})}{\Gamma(1 + \frac{j}{\alpha})} \\ &\times e^{-\frac{b}{\alpha}(m+n)} (b/\alpha)^{\frac{m}{2\alpha}(m-1-2b) + \frac{n}{2\alpha}(n-1-2b) + \frac{m+n}{2}}. \end{aligned} \quad (5.8)$$

Together with (5.7) we thus complete the proof. \square

For β even, a transition from the limiting correlation functions near the spectrum singularity to the soft-edge correlation immediately follows from Theorem 5.1.

Corollary 5.2. *Assume that β is even and $b \in \mathbb{R}$. Let $R_{b,1}^{(k,\beta/2)}(y)$ be defined by (3.10), we have*

$$\begin{aligned} \lim_{b \rightarrow \infty} \left(\frac{8b}{\beta}\right)^{k/3} R_{b,1}^{(k,\beta/2)}\left(-\frac{4b}{\beta} + \left(\frac{8b}{\beta}\right)^{1/3} \xi_1, \dots, -\frac{4b}{\beta} + \left(\frac{8b}{\beta}\right)^{1/3} \xi_k\right) = \\ a_k(\beta) |\Delta_k(\xi)|^\beta \text{Ai}_{\beta k}^{(\beta/2)}(\xi_1, \dots, \xi_1, \dots, \xi_k, \dots, \xi_k) \end{aligned} \quad (5.9)$$

where in the argument of $\text{Ai}_{\beta k}^{(\beta/2)}$ each ξ_j ($j = 1, \dots, k$) occurs β times, and $a_k(\beta)$ is defined by (4.41).

Proof. Let $\alpha = \beta/2$. Combine the definition of $S_b^{(\alpha)}(x)$ and Theorem 5.1, one easily completes the proof. \square

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REFERENCES

- [1] G. Akemann, P.H. Damgaard, U. Magnea, S.M. Nishigaki, *Universality of random matrices in the microscopic limit and the Dirac operator spectrum*, Nucl. Phys. B 487(3)(1997), 721–738.
- [2] G. Akemann, P.H. Damgaard, U. Magnea, S.M. Nishigaki, *Multicritical microscopic spectral correlators of Hermitian and complex matrices*, Nucl. Phys. B 519(3)(1998), 682–714.

- [3] G. Akemann and Y. V. Fyodorov, *Universal random matrix correlations of ratios of characteristic polynomials at the spectral edges*, Nucl. Phys. B 664 (2003), 457–476.
- [4] K. Aomoto, *Scaling limit formula for 2-point correlation function of random matrices*, Adv. Stud. Pure Math. 16: conformal field theory and solvable lattice models, 1–15.
- [5] J. Baik, P. Deift, and E. Strahov, *Products and ratios of characteristic polynomials of random Hermitian matrices*, J. Math. Phys. 44 (2003), 3657–3670.
- [6] T. H. Baker and P. J. Forrester, *The Calogero-Sutherland model and generalized classical polynomials*, Commun. Math. Phys. 188 (1997), 175–216.
- [7] A. Borodin and E. Strahov, *Averages of Characteristic Polynomials in Random Matrix Theory*, Commun. Pure and Appl. Math. 59 (2006), 161–253.
- [8] G. Borot and A. Guionnet, *Asymptotic expansion of β matrix models in the one-cut regime*, Commun. Math. Phys. 317 (2013), Issue 2, 447–483.
- [9] P. Bourgade, L. Erdős, and H.-T. Yau, *Bulk Universality of General β -Ensembles with Non-convex Potential*, J. Math. Phys. 53 (2012), 095221.
- [10] P. Bourgade, L. Erdős, and H.-T. Yau, *Universality of General β -Ensembles*, Duke Math. J. 163, No.6 (2014), 1127–1190.
- [11] P. Bourgade, C.-P. Hughes, A. Nikeghbali, M. Yor, *The characteristic polynomial of a random unitary matrix: a probabilistic approach*, Duke Math. J. 145 (2008), no. 1, 45–69.
- [12] P. Bourgade, A. Nikeghbali, and A. Rouault, *Circular Jacobi Ensembles and Deformed Verblunsky Coefficients*, Int. Math. Res. Notices 2009: 23 (2009), 4357–4394.
- [13] P. Bourgade, J. Najnudel and A. Nikeghbali, *A unitary extension of virtual permutations*, Int. Math. Res. Notices 2013: 18 (2013), 4101–4134.
- [14] R. Chhaibi, J. Najnudel and A. Nikeghbali, *A limiting random analytic function related to the CUE*, arXiv:1403.7814.
- [15] P. Desrosiers, *Duality in random matrix ensembles for all β* , Nucl. Phys. B 817 (2009), 224–251.
- [16] P. Desrosiers and P. J. Forrester, *Hermite and Laguerre β -ensembles: Asymptotic corrections to the eigenvalue density*, Nucl. Phys. B 43 (2006), 307–332.
- [17] P. Desrosiers, D.-Z. Liu, *Asymptotics for products of characteristic polynomials in classical β -ensembles*, Constr. Approx. 39, no. 2 (2014), 273–322.
- [18] P. Desrosiers, D.-Z. Liu, *Scaling Limits of Correlations of Characteristic Polynomials for the Gaussian β -Ensemble with External Source*, Int. Math. Res. Notices 2014, doi:10.1093/imrn/rnu039, 31 pp.
- [19] A. Dubbs, A. Edelman, *The beta-MANOVA ensemble with general covariance*, Random Matrices: Theory and Appl. 3, No. 1 (2014), 1450002, 16pp.
- [20] A. Dubbs, A. Edelman, P. Koev, *The beta-Wishart ensemble*, J. Math. Phys. 54, 083507 (2013), 20pp.
- [21] L. Dumaz and B. Virág, *The right tail exponent of the Tracy-Widom beta distribution*, Ann. I. H. P. Prob. Stat. 49 No. 4(2013), 915C-933.
- [22] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. 43 (2002), 5830–5847.
- [23] I. Dumitriu and A. Edelman, *Eigenvalues of Hermite and Laguerre ensembles: large beta asymptotics*, Ann. I. H. P. Prob. Stat. 41 (2005), 1083–1099.
- [24] I. Dumitriu and A. Edelman, *Global spectrum fluctuations for the β -Hermite and β -Laguerre ensembles via matrix models*, J. Math. Phys. 47 (2006), 063302 1–36.
- [25] I. Dumitriu and P. Koev, *Distributions of the extreme eigenvalues of beta-Jacobi random matrices*, SIAM J. Matrix Anal. Appl. 30 (2008), 1–6.
- [26] A. Edelman and B. D. Sutton, *From Random Matrices to Stochastic Operators*, J. Stat. Phys. 127 (2007), 1121–1165.
- [27] P. J. Forrester, *Selberg correlation integrals and the $1/r^2$ quantum many-body system*, Nucl. Phys. B 388 (1992), 671–699.
- [28] P. J. Forrester, *Exact results and universal asymptotics in the Laguerre random matrix ensemble*, J. Math. Phys. 35 (1993), 2539–2551.
- [29] P. J. Forrester, *Log-gases and Random Matrices*, London Mathematical Society Monographs 34, Princeton University Press (2010).
- [30] P. J. Forrester, *Probability densities and distributions for spiked and general variance Wishart β -ensembles*, Random Matrices: Theory and Appl. 2, No. 4 (2013), 1350011, 19pp.

- [31] P. J. Forrester, T. Nagao, *Correlations for the Cauchy and generalized circular ensembles with orthogonal and symplectic symmetry*, J. Phys. A: Math. Gen. 34 (2001), 7917–7932.
- [32] D. Holcomb, G. R. Moreno Flores, *Edge scaling of the β -Jacobi ensemble*, J. Stat. Phys. 149 (2012), 1136–1160.
- [33] L. K. Hua, *Loo Keng Hua Collected Works: Function theory of several complex variables I*, Science Press, Beijing, 2010 (in Chinese); *Harmonic analysis of functions of several complex variables in the classical domains*, American Mathematical Society, Providence, RI, 1963.
- [34] C.P. Hughes, J.P. Keating, and N. O’Connell, *On the Characteristic Polynomial of a Random Unitary Matrix*, Comm. Math. Phys. 220(2) (2001), 429–451.
- [35] A. T. James, *Distributions of matrix variates and latent roots derived from normal samples*, Ann. Math. Statist. 35(2)(1964), 475–501.
- [36] T. Jiang, *Limit theorems for beta-Jacobi ensembles*, Bernoulli 19(3)(2013), 1028–1046.
- [37] N. Jokela, M. Järvinen and E. Keski-Vakkuri, *Electrostatics of Coulomb gas, lattice paths and discrete polynuclear growth*, J. Phys. A: Math. Theor. 43 (2010), 425006 (32pp).
- [38] J. Kaneko, *Selberg integrals and hypergeometric functions associated with Jack polynomials*, SIAM J. Math. Anal. 24 (1993), 1086–1110.
- [39] J. P. Keating and N. C. Snaith, *Random matrix theory and $\zeta(1/2 + it)$* , Commun. Math. Phys. 214 (1) (2000), 57–89.
- [40] R. Killip, *Gaussian Fluctuations for β Ensembles*, IMRN (2008) rnn007, 1–19.
- [41] R. Killip and I. Nenciu, *Matrix Models for Circular Ensembles*, IMRN (2004), 2665–2701.
- [42] F. Knop, S. Sahi, *A recursion and a combinatorial formula for Jack polynomials*, Invent. math. 128 (1997), 9–22.
- [43] A.B.J. Kuijlaars, M. Vanlessen, *Universality for Eigenvalue Correlations at the Origin of the Spectrum*, Commun. Math. Phys. 243 (2003), 163–191.
- [44] I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (2nd ed.), Oxford University Press Inc, New York, 1995.
- [45] K. Maples, J. Najnudel and A. Nikeghbali, *Limit operators for circular ensembles*, arXiv:1304.3757.
- [46] S. Matsumoto, *Moments of characteristic polynomials for compact symmetric spaces and Jack polynomials*, J. Phys. A 40 (2007), 13567–13586.
- [47] M. L. Mehta, *Random Matrices* (3rd ed), Singapore: Elsevier, 2004.
- [48] T. Nagao, *Universal correlations near a singularity of random matrix spectrum*, J. Phys. Soc. Japan 64 no.10 (1995), 3675–3681.
- [49] T. Nagao and K. Slevin, *Nonuniversal correlations for random matrix ensembles*, J. Math. Phys. 34(5)(1993), 2075–2085.
- [50] J. Najnudel, A. Nikeghbali, A. Rouault, *Limit theorems for orthogonal polynomials related to circular ensembles*, arXiv:1302.6501.
- [51] J. A. Ramírez, B. Rider, *Diffusion at the Random Matrix Hard Edge*, Commun. Math. Phys. 288 (2009), 887–906.
- [52] J. A. Ramírez, B. Rider, and B. Virág, *Beta ensembles, stochastic Airy spectrum, and a diffusion*, J. Amer. Math. Soc. 24 (2011), 919–944.
- [53] J. A. Ramírez, B. Rider and O. Zeitouni, *Hard edge tail asymptotics*, ECP 16 (2011), 741–752.
- [54] M. Rösler, *Generalized Hermite Polynomials and the Heat Equation for Dunkl Operators*, Commun. Math. Phys. 192(1998), 519–541.
- [55] Z. Su, *Circular β ensembles, CMV representation, characteristic polynomials*, Sci. China Ser. A 52, no.7 (2009), 1467–1477.
- [56] Z. G. Su, *On the second-order correlation of characteristic polynomials of Hermite β ensembles*, Stat. Proba. Letters. 80 (2010), 1500–1507.
- [57] R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. 77, 76–115.
- [58] B. Valkó and B. Virág, *Continuum limits of random matrices and the Brownian carousel*, Inventiones Mathematicae Vol. 177 (2009), 463–508.
- [59] B. Valkó and B. Virág, *Large gaps between random eigenvalues*, Annals of Probab. 38 (2010), 1263–1279.
- [60] D. Wang, *The largest eigenvalue of real symmetric, Hermitian and Hermitian self-dual random matrix models with rank one external source, part I*, J. Stat. Phys. 146 (2012), no. 4, 719–761.

- [61] Z. Yan, *A class of generalized hypergeometric functions in several variables*, Can. J. Math. **44** (1992), 1317–1338.

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